

## Centripetal Motion and Gravity

Consider a satellite orbiting the Earth. It does so in a very nearly circular path, because:

- The spacecraft attempts to follow a “horizontal” straight-line path that is tangent to the curvature of the Earth.
- The Earth exerts a force of gravity on the satellite, pulling it downward much like an object that is thrown horizontally from a cliff.
- On a large scale, the Earth’s surface is not flat; it curves away so that  $\mathbf{F}_g$  always acts *radially*, i.e., into the center of the planet
- The balance between the inertia of the satellite attempting to go “straight” and the force of gravity  $\mathbf{F}_g$  that pulls perpendicular to its motion causes the satellite to travel in its circular path around the planet, maintaining its elevation above Earth’s surface. Note though:
  - if the satellite were to travel too fast, it would pull away from the Earth and no longer follow a circular path.
  - if the satellite were travelling too slow, it would also no longer follow a circular path; in this case, it would eventually fall into the Earth.

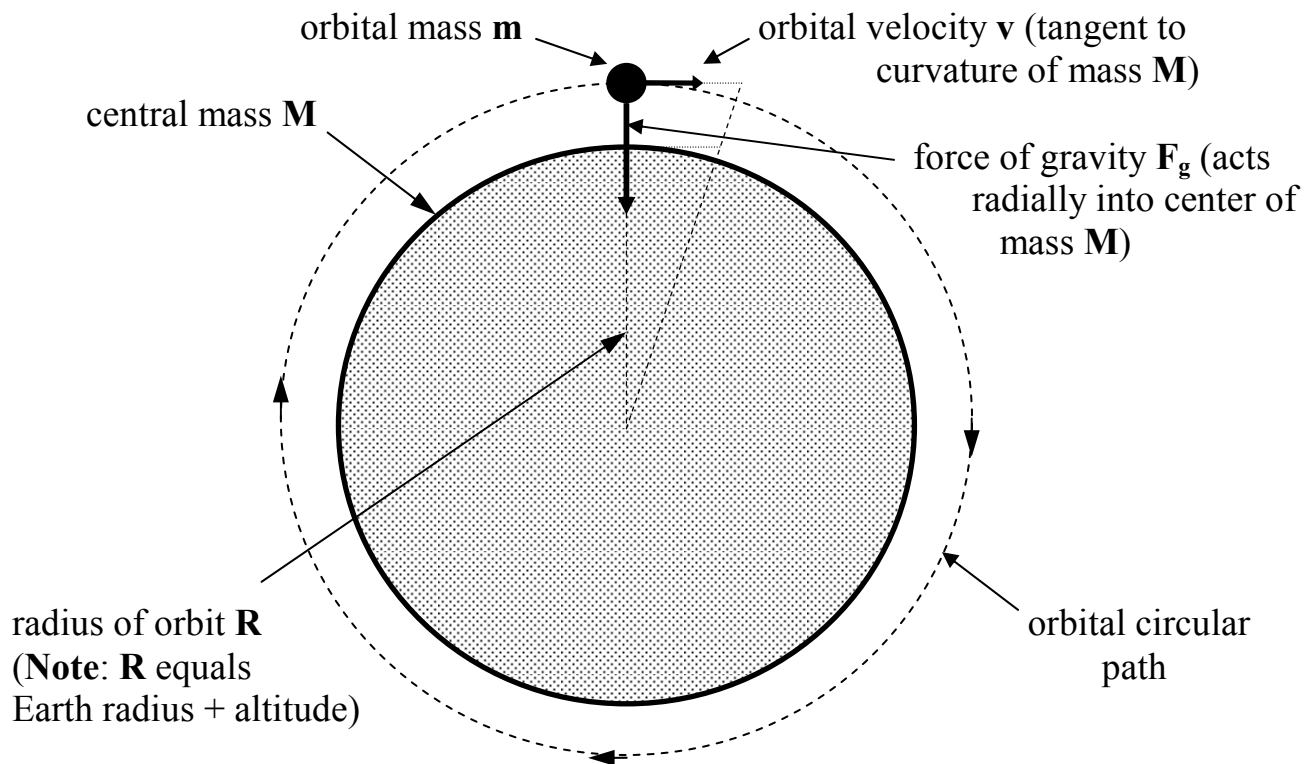
Stable, relatively circular orbits occur commonly in space. For example:

- natural satellites (called ‘moons’) orbit a central mass planet.
- planets orbit a central mass star (like the Sun).
- stars orbit the central matter of galaxies.

In each case, the *central mass* ‘ $\mathbf{M}$ ’ supplies the force of gravity to maintain the circular orbit. The net force on the orbiting mass is a centripetal force, so we can state that, for any stable orbiting body:

$$\mathbf{F}_c = \mathbf{F}_g$$

The next page illustrates the nature of a stable orbit.



The velocity for a stable satellite orbit can be determined in the following way:

- In this case, gravity provides the centripetal force; that is,  $F_c = F_g$

- Substitute in the formulas: 
$$\frac{mv^2}{R} = \frac{GMm}{R^2}$$

→ cancel out orbital mass 'm' to get 
$$\frac{v^2}{R} = \frac{GM}{R^2}$$

- this tells us that  $a_c = g$  → the centripetal acceleration of the satellite must equal the gravitational field strength at that altitude!

- note that 'R' also cancels once, so that 
$$v = \sqrt{\frac{GM}{R}}$$

From these equations we see the following relationships:

- $F_g \propto v^2$
- $F_g \propto 1/R$
- $v \propto \frac{1}{\sqrt{R}}$

**Example 6:**

- (a) Determine the stable parking orbit velocity for a surveying satellite located 230 km above the moon's surface.**
- (b) If that orbital radius were reduced by one-tenth, by what factor would the orbiting speed increase?**

**(see Gravitation Ex 6 for answer)**

## Centripetal Force

Examine Newton's 1<sup>st</sup> and 2<sup>nd</sup> Laws of motion:

- Newton's 1<sup>st</sup> Law of Inertia states that, in the absence of an external, unbalanced force, any object will keep on doing what it's already doing. This means that a stationary object will remain stationary, and an object travelling at speed in one direction will continue on in the same direction at constant speed.
- The 2<sup>nd</sup> law explains that if an unbalanced or net force does exist on an object, that object will accelerate, with the formula relating the two quantities being  $\mathbf{F}_{\text{Net}} = \mathbf{ma}$ .

In Physics 11, you learned that when an unbalanced force acts on the object and the object accelerates, it must either speed up or slow down.

But hold on: an unbalanced force can also cause an object to simply change direction. For example, if a hockey puck slides along the ice at a constant speed and hits the goal post, the force that the post exerts on the puck causes the puck to change direction. Ignoring the friction of puck-on-ice (which is pretty small), the collision between puck-and-post is the only significant force (and therefore the *net* force) affecting the puck's motion.

The centripetal acceleration of an object travelling in a circle at constant speed is caused by an unbalanced force. In this case, the net force acts *perpendicular* to the object's motion, causing it to veer out of its straight path and into a circular path. In other words, a *centripetal force* causes centripetal acceleration, or  $\mathbf{F}_c = \mathbf{ma}_c$ .

To calculate centripetal force, start with  $\mathbf{F}_{\text{Net}} = \mathbf{F}_c$  for any object moving in a circular motion. If  $\mathbf{F}_{\text{Net}} = \mathbf{ma}$ :

→ substitute  $\mathbf{a}_c = \frac{v^2}{r}$  to get  $\mathbf{F}_c = \mathbf{m} \frac{v^2}{r}$

→ or, substitute  $\mathbf{a}_c = \frac{4\pi^2 r}{T^2}$  to get  $\mathbf{F}_c = \mathbf{m} \frac{4\pi^2 r}{T^2}$

Note that these two formulas aren't on the formula sheet, but don't panic: simply add an '**m**' to the centripetal acceleration formulas.

Some kinds of forces that can act as a centripetal force include:

- the friction force that holds a child's feet to the platform of a merry-go-round while going round-and-round in a circle;
- the tension force of a rope that allows a tether ball to go around a pole without flying out.
- The gravitational force used by the Sun to keep the Earth in orbit, rather than heading out into space (and beyond).

**Example #3: A 750 kg car travelling at 18.0 m/s comes to a sharp turn in the road, where the radius of the curve is 136 m.**

- a) **Find the centripetal acceleration and force acting on the car as it begins the turn.**
- b) **If the coefficient of static friction between tires and road is  $\mu = 0.254$ , will the car be able to complete the turn at this speed without sliding off the road?**

**(see Circular Motion Ex 3 for answer)**

## Analysis of Force Vectors in a Vertical Circle.

There are numerous examples of vertical circular motion. Some obvious ones include:

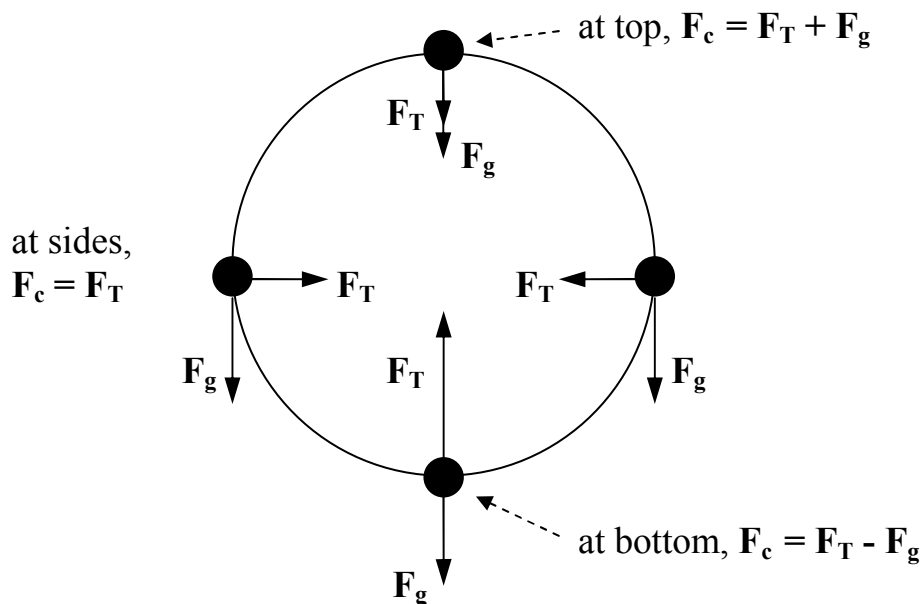
- roller coasters and toy cars doing loop-the-loops;
- twirling a mass attached to a cord, much like a 360° pendulum;
- riding a Ferris wheel;
- driving a vehicle over a round hill.

For these and all vertical circular motion problems, the net force is not always centripetal, and is therefore not dealt with in Physics 12. For our purposes we will only consider those forces which affect the *net centripetal force*. This means we can only analyze forces acting on a mass at specific positions in the vertical circle.

Objects that travel in vertical, circular paths typically have a minimum of two forces acting on them. By examining free-body diagrams of these forces, the net centripetal force can be determined mathematically.

### **A. Bob attached to a string.**

Examine the forces on a bob going in a vertical circle; note that forces pointing **into** the circle center are *positive*, while forces pointing **away** from the center are *negative*. As well, any *tangential* force has no effect on the net centripetal force.



The tension in the string  $F_T$  that connects the bob to the center must supply the force needed to provide the net centripetal force and to support the bob's weight.

At the bottom, the tension in this string is  $F_T = F_c + F_g$

At the side,  $F_g$  is tangential, so  $F_T = F_c$

At the top, the weight contributes towards  $F_c$ :  $F_T = F_c - F_g$

**Example #4:** A 0.90 kg mass attached to a cord is whirled in a vertical circle of radius 2.5 m.

- a) Find the tension in the cord at the top of the circle if the speed of the mass is 8.7 m/s.
- b) Find the tension in the cord at the bottom of the circle if the speed is maintained at 8.7 m/s.

(see Circular Motion Ex 4 for answer)

**Note:** at the top of these loops, there is a certain minimum speed, sometimes called the *critical velocity*, that will just keep the mass going around the loop on the track. It is the speed at which the track exerts no normal/tension force, so that the needed centripetal force equals the weight of the mass, or  $F_c = F_g$ . If the mass goes slower, then  $F_c < F_g$  and the mass will *fall*.

**Example #5:** The same system above is now whirled at a slower rate.

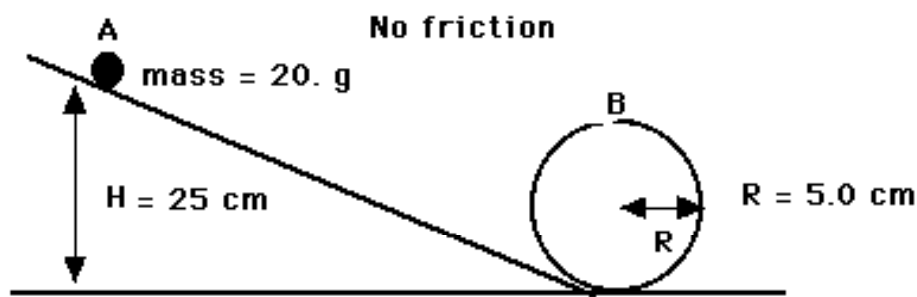
- a) What minimum speed must it have at the top of the circle so as not to fall from the circular path?
- b) At the speed in (a) and neglecting any friction, how fast will the object be going at the bottom of the circle?
- c) What is the tension in the cord at the bottom at this speed?

(see Circular Motion Ex 5 for answer)

### B. A vertical loop of track (e.g., on a roller coaster).

A vehicle going around a track loop will have the normal force of the track ( $F_N$ ) acting on the mass and contributing to the centripetal force in the same way as  $F_T$  does. Simply substitute  $F_N$  in for  $F_T$  in the above equations.

**Example #6:** A 20. gram steel ball-bearing on a rail rolls from rest at point A, as shown below.



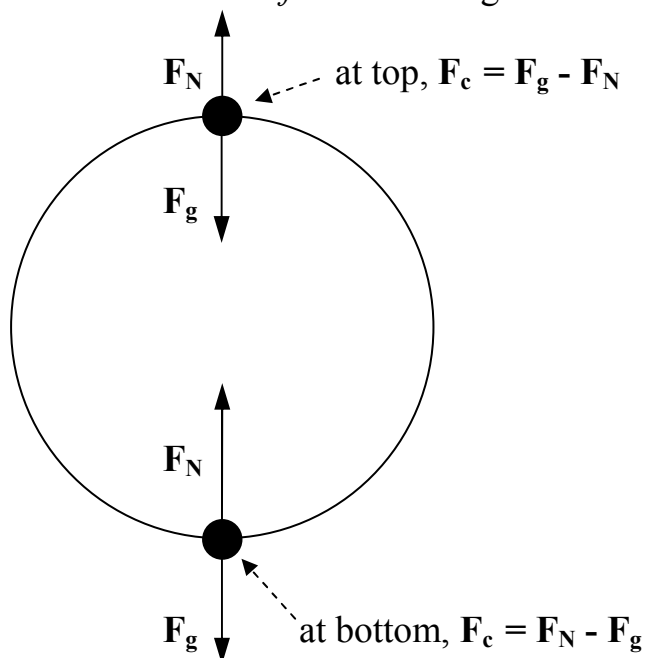
Assuming negligible friction, if  $h = 0.25$  m and  $R = 0.050$  m,

- what is the speed of the bearing at point B?
- what normal force must the rail exert on the bearing at B?

(see Circular Motion Ex 6 for answer)

### C. The ferris wheel.

In this situation, the forces are arranged slightly differently; note in particular the way in which the *normal force* is arranged at the top:



At these two locations, the normal force acts up to counteract gravity, but each equation is different (remember, pointing *in* is positive).

As well, at the top there is a *maximum* critical speed where the person will just remain in her seat, without flying off tangentially. The normal force is 0, so once again, at this speed:  $F_c = F_g$



Note that this analysis also works for vehicles driving up and over a hill.

**Example #7: A 62 kg student drives his 450 kg car at 25 m/s up towards the top of a hill of radius 70. m.**

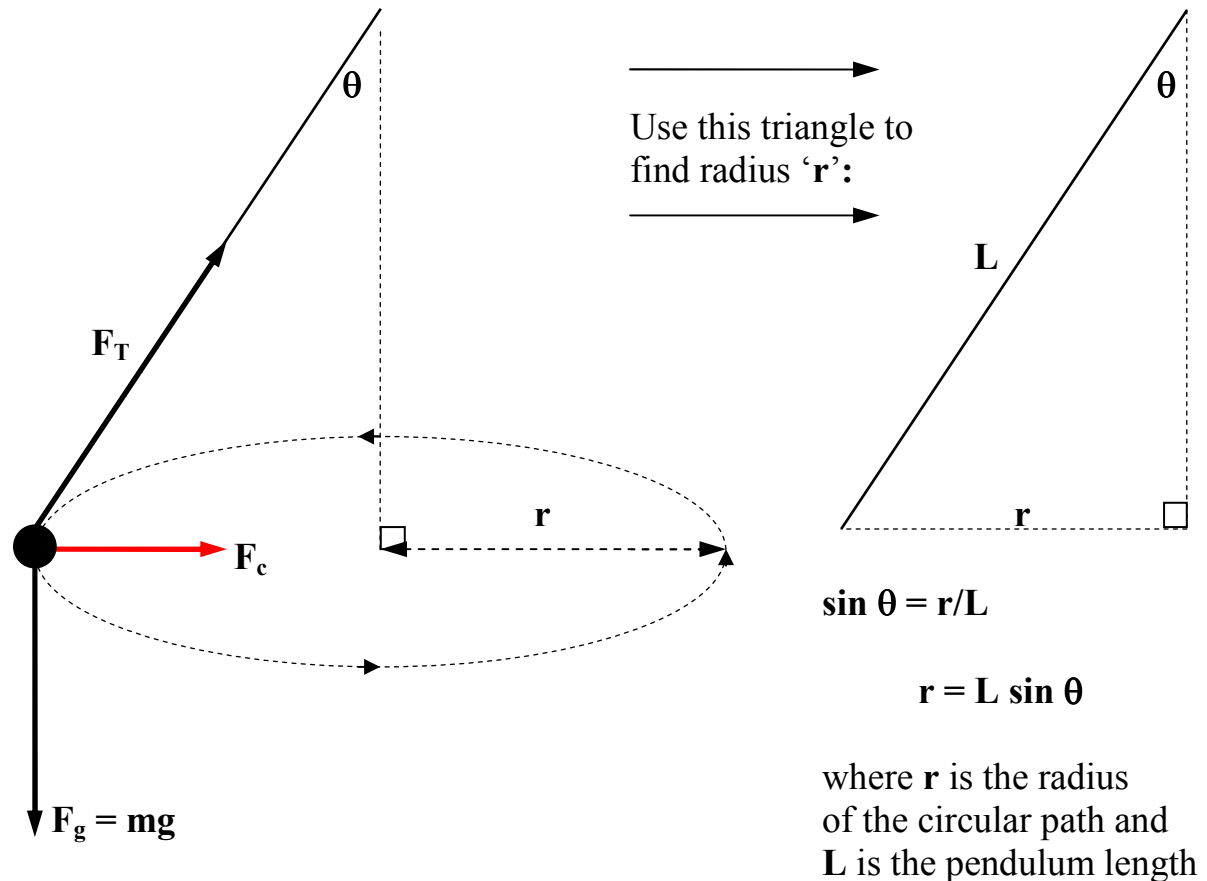
- a) What normal force will the driver's seat exert on him at the top of the hill?**
- b) How fast can he drive his car over the hill without being airborne?**

**(see Circular Motion Ex 7 for answer)**

## Analysis of Force Vectors in Two Dimensions.

Now we examine circular motion caused by forces that aren't acting on the same plane as the circle.

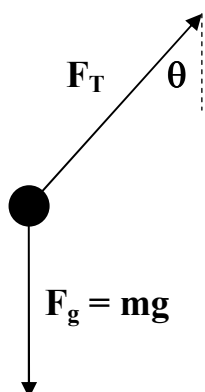
### A: The Conical Pendulum



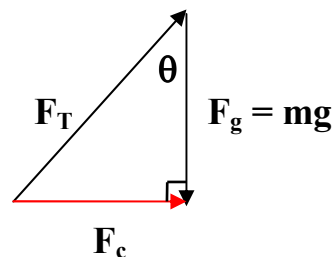
Examine the above diagram and make note of the vectors drawn.

In this situation, there are only two forces acting on the pendulum bob: the tension force  $F_T$  and the weight of the bob  $F_g$ . The centripetal force  $F_c$  is actually a *net force*, the resultant sum of the two previous forces mentioned, as shown on the next page:

The free-body diagram:



→ add these vectors  
and draw the  
resultant  $F_c$  →



A right triangle is the result. Use trigonometry, along with

$$F_c = m \frac{4\pi^2 r}{T^2} \quad \text{and} \quad F_c = m \frac{v^2}{r}$$

to find tension  $F_T$ , speed  $v$ , period  $T$ , etc.

**Example #8:** A 140 g ball is fastened to one end of a 0.24 m string, and the other end is whirled in a horizontal conical pendulum. Find:

- the speed of the ball in its circular path;
- the tension in the string that makes an angle of  $30^\circ$  to the vertical.

(see Circular Motion Ex 8 for answer)

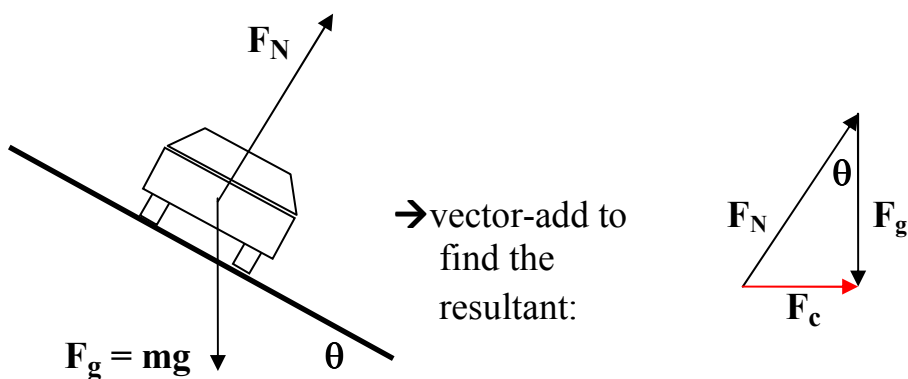
### **B: Banked Curves**

The purpose of banking a roadway, especially on an oval racetrack where cars are travelling at high speeds, is so that the bank itself can provide all the turning force. In this way, no friction is needed and accidents are less likely to occur.

In other words, for a particular speed, the car can steer itself around the corner and never miss a turn because of ice. This is the *null speed*, the speed at which the curve is said to have “ideal banking” or “proper” banking.

Note that the equation for null speed can be derived. At this speed, there are only two forces (the weight and the normal force) which add together to supply the centripetal force ( $F_c$ ):

Free-body diagram of car on a curved bank, coming towards you (out-of-page) and turning to the right:



To the right, the sum of the two forces produces a right triangle with resultant  $F_c$ , just as in the conical pendulum. From this diagram, we see that:

$$\tan \theta = \frac{F_c}{mg} \quad \text{where} \quad F_c = m \frac{v^2}{r} \quad \text{and 'r' is the radius of the track's curve.}$$

$$\rightarrow \text{substitute and cancel to obtain} \quad \tan \theta = \frac{v^2}{rg}$$

Note that this derivation is based on the null speed only (i.e. no friction).

**Example #9:** A curve of 30 m radius is banked so that a car may make a turn at a speed of 13 m/s without depending on friction at all. What is the slope of the curve?

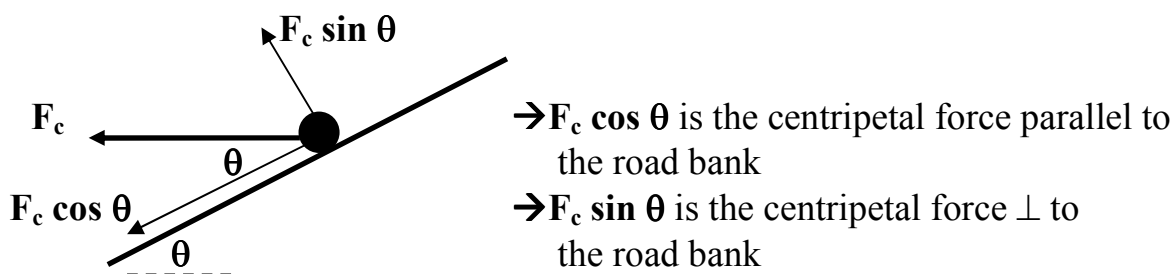
(see Circular Motion Ex 9 for answer)

What would occur if you were driving a car around an “ideally” banked corner at a speed:

- a) below the null speed?
  - if the car is travelling too **slowly**, the normal force of the track will not be great enough to hold the car in a circular path, so the car will start to slide down the bank;
  - therefore, friction must act *up*-bank to keep the vehicle in circular motion.
- b) greater than the null speed?
  - if the car is travelling too **quickly**, its momentum would cause the car to start sliding up the bank;
  - therefore, a friction force must act *down*-bank to prevent this motion and keep the vehicle moving in a circular path.

In either case, there are too many vectors to create a triangle, and is therefore beyond the scope of this course. However, if you're finding this material much too simple and need a challenge, read on!

Examine force components both parallel and perpendicular to the surface of the bank.



Examining the perpendicular forces,

- $F_c \sin \theta$  is the unbalanced force which supplies the perpendicular component of  $\mathbf{a}_c$ .
- The normal force ( $F_N$ ) acts positively to supply this net force, while the perpendicular component of the car's weight ( $F_g \cos \theta$ ) acts against it, so that:

$$F_c \sin \theta = F_N - F_g \cos \theta \quad (1)$$

Examining the parallel forces,

- we see that inertia would carry the mass up-slope as it rounds the bank; an unbalanced force,  $F_c \cos \theta$  exists which keeps this from happening.
- The parallel component of the car's weight ( $F_g \sin \theta$ ) will act *positively* down-slope to supply this net force. As well,
  - if the car is travelling too slowly, friction will act up-slope, *negatively*, so that:

$$F_c \cos \theta = F_g \sin \theta - F_f \quad (2)$$


- if the car is travelling too quickly, friction will act down-slope, *positively*, so that:

$$F_c \cos \theta = F_g \sin \theta + F_f \quad (3)$$


Use either equation (1) and (2) or (3) by substituting in  $F_c = m \frac{4\pi^2 r}{T^2}$  and  $F_c = m \frac{v^2}{r}$ , as well as  $F_f = \mu F_N$  to solve for unknown values.

Remember, this is a university-style problem, so if you're having difficulty with it, give it a pass, since it is unlikely to show up on the final exam.

**Example #10:** The radius of a velodrome curve is 40 m and the banked angle is 15°. If  $\mu=0.20$ , what is the maximum speed at which a cyclist can take this curve without slipping?

(see Circular Motion Ex 10 for answer)

**Example #1: A child stands on a merry-go-round that is spun around with a period of 2.3 s.**

- a) If the child is 0.65 m from the center of the merry-go-round, what is her acceleration?
- b) If she then moves to a distance of 1.4 m from the center, at what speed will she be travelling?

$$a) \quad a_c = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 (.65)}{2.3^2}$$

$a_c = 4.9 \text{ m/s}^2$

$$b) \quad v = \frac{2\pi r}{T} = \frac{2\pi (1.4)}{2.3 \rightarrow \text{constant}}$$

$v = 3.8 \text{ m/s}$

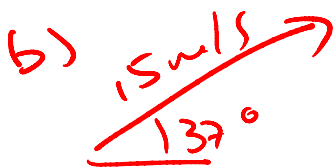
**Example #2:** In the Olympics, the Hammer Throw competition involves athletes spinning a heavy weight on a cord around several times before releasing the weight at an angle to the ground in order for it to travel a maximum distance. If one such weight of mass 15 kg was spun 4.5 times in 3.2 s in a circle of radius 1.7 m, then released at an angle of  $37^\circ$  to the ground,

- what was the average speed and acceleration of the weight as it was spun?
- how far was it flung, assuming it landed at about the same height it was released?

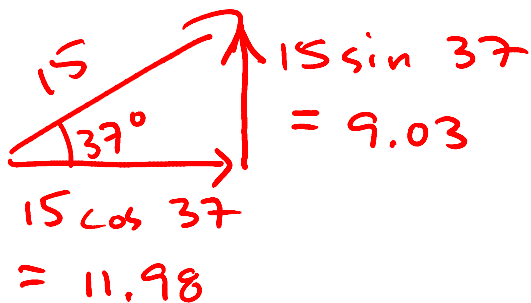
$$a) \quad T = \frac{3.2}{4.5} = 0.711 \text{ s}$$

$$v = \frac{2\pi r}{T} = \frac{2\pi(1.7)}{0.711} \quad \boxed{v = 15 \text{ m/s}}$$

$$a = \frac{v^2}{r} = \frac{15^2}{1.7} \quad \boxed{a_c = 1.3 \times 10^2 \text{ m/s}^2}$$



$\Rightarrow$  use kinematics to solve



vert:

$$v = v_0 + at$$

$$-9.03 = 9.03 + -9.8t$$

$$t = 1.84 \text{ s (up + down)}$$

hor:

$$d = v_w t = 11.98(1.84) \quad \boxed{d = 22 \text{ m}}$$



**Example #3:** A 750 kg car travelling at 18.0 m/s comes to a sharp turn in the road, where the radius of the curve is 136 m.

- Find the centripetal acceleration and force acting on the car as it begins the turn.
- If the coefficient of static friction between tires and road is  $\mu = 0.254$ , will the car be able to complete the turn at this speed without sliding off the road?

$$a) \quad a_c = \frac{v^2}{r} = \frac{18^2}{136} \quad \boxed{a_c = 2.38 \text{ m/s}^2}$$

$$F_c = ma_c = 750(2.38) \quad \boxed{F_c = 1.79 \times 10^3 \text{ N}}$$

b) Use  $\mu_s = .254$  to find max.  $F_f$  available before slippage occurs

$$F_f = \mu F_N = .254(750)(9.8)$$

$$F_f = 1.9 \times 10^3 \text{ N}$$

→ since  $F_c$  is provided by friction, there is more than enough to do the job, so yes, the turn will be completed

**Example #4:** A 0.90 kg mass attached to a cord is whirled in a vertical circle of radius 2.5 m.

- a) Find the tension in the cord at the top of the circle if the speed of the mass is 8.7 m/s.  
b) Find the tension in the cord at the bottom of the circle if the speed is maintained at 8.7 m/s.

a)



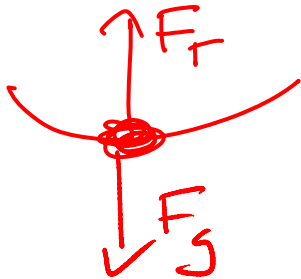
$$F_c = F_g + F_T$$

$$F_T = F_c - F_g$$
$$= \frac{mv^2}{r} - mg$$

$$= .90 \left[ \frac{8.7^2}{2.5} - 9.8 \right]$$

$$F_T = 18 \text{ N}$$

b)



$$F_c = F_T - F_g$$

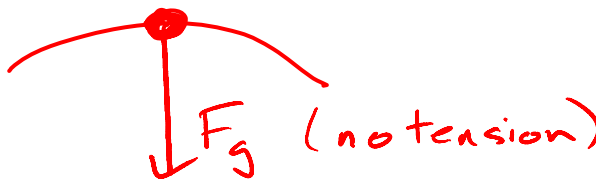
$$F_T = F_c + F_g$$

$$= .90 \left[ \frac{8.7^2}{2.5} + 9.8 \right]$$

$$F_T = 36 \text{ N}$$

**Example #5:** The same system above is now whirled at a slower rate.

- What minimum speed must it have at the top of the circle so as not to fall from the circular path?
- At the speed in (a) and neglecting any friction, how fast will the object be going at the bottom of the circle?
- What is the tension in the cord at the bottom at this speed?



$$F_c = F_g$$

$$\frac{mv^2}{r} = mg$$

$$v = \sqrt{rg} = \sqrt{2.5(9.8)}$$

$$v = 4.9 \text{ m/s} \quad (4.95)$$

b) Use conservation of energy to solve.

→ At top of circle,  $E_T = E_p + E_k$

$$E_T = (.90)(9.8)(5.0)^{(2 \times r)} + \frac{1}{2}(.90)(4.95)^2$$

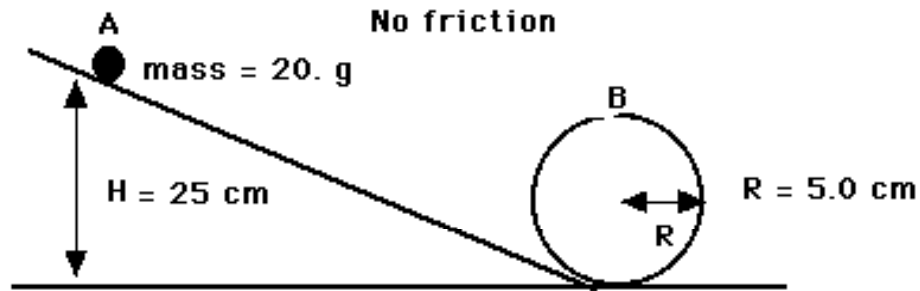
$$= 55.1 \text{ J.}$$

→ At bottom of circle,  $E_T = E_k$

$$55.1 = \frac{1}{2}(.9)v^2$$

$$v = 11 \text{ m/s}$$

Example #6: A 20. gram steel ball-bearing on a rail rolls from rest at point A, as shown below.



Assuming negligible friction, if  $h = 0.25$  m and  $R = 0.050$  m,

- what is the speed of the bearing at point B?
- what normal force must the rail exert on the bearing at B?

a) Use cons. of energy :

$$\text{at "A", } E_T = E_P = mgh \\ = .02(9.8)(.25) = 0.049 \text{ J}$$

$$\text{at "B", } E_T = E_P + E_K \text{ (2+r)} \\ .049 = .020(9.8)(.10) + \frac{1}{2}(.020)v^2$$

$$v = 1.7 \text{ m/s}$$

b)



$$F_c = F_N + F_g$$

$$F_N = F_c - F_g = \frac{mv^2}{r} - mg$$

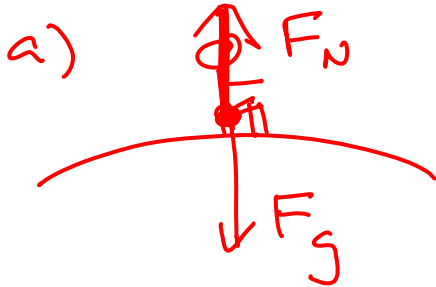
$$= .02 \left[ \frac{1.7^2}{.05} - 9.8 \right]$$

$$F_N = 0.98 \text{ N}$$

Example #7: A 62 kg student drives his 450 kg car at 25 m/s up towards the top of a hill of radius 70. m.

a) What normal force will the driver's seat exert on him at the top of the hill?

b) How fast can he drive his car over the hill without being airborne?



$$F_c = F_g - F_N$$

$$F_N = F_g - F_c = mg - \frac{mv^2}{r}$$

$$= 62 \left[ 9.8 - \frac{25^2}{70} \right]$$

$$F_N = 54 \text{ N}$$

b) at max speed,  $F_N = 0$

so  $F_c = F_g$

$$\frac{mv^2}{r} = mg$$

$$v = \sqrt{rg}$$

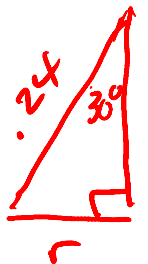
$$v = \sqrt{70(9.8)}$$

$$v = 26 \text{ m/s}$$

**Example #8:** A 140 g ball is fastened to one end of a 0.24 m string, and the other end is whirled in a horizontal conical pendulum. Find:

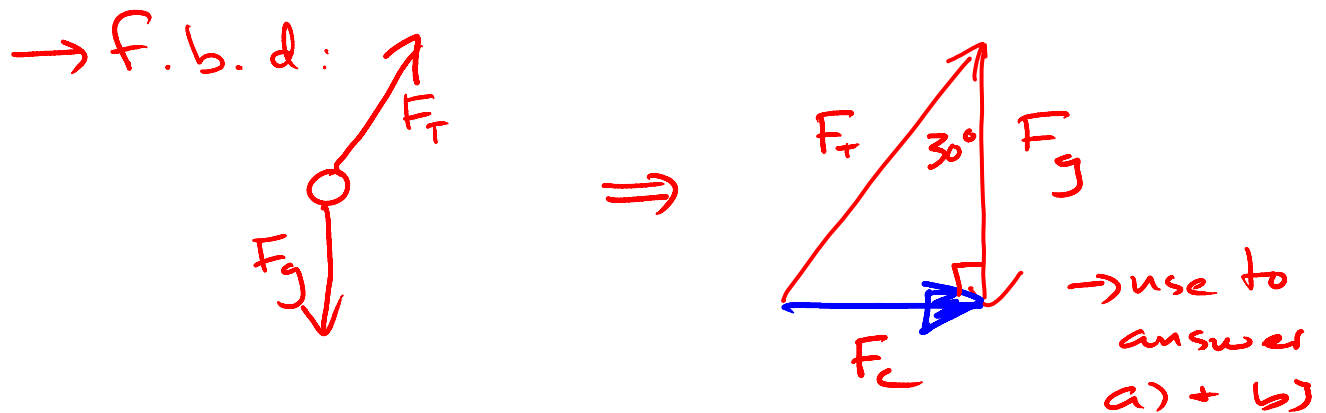
- the speed of the ball in its circular path;
- the tension in the string that makes an angle of  $30^\circ$  to the vertical.

→ find radius:



$$r = 0.24 \sin 30$$

$$= \underline{\underline{0.12 \text{ m}}}$$



$$a) \frac{F_c}{F_g} = \tan \theta \quad \frac{mv^2}{r} = mg \tan \theta$$

$$v = \sqrt{0.12 (9.8) \tan 30}$$

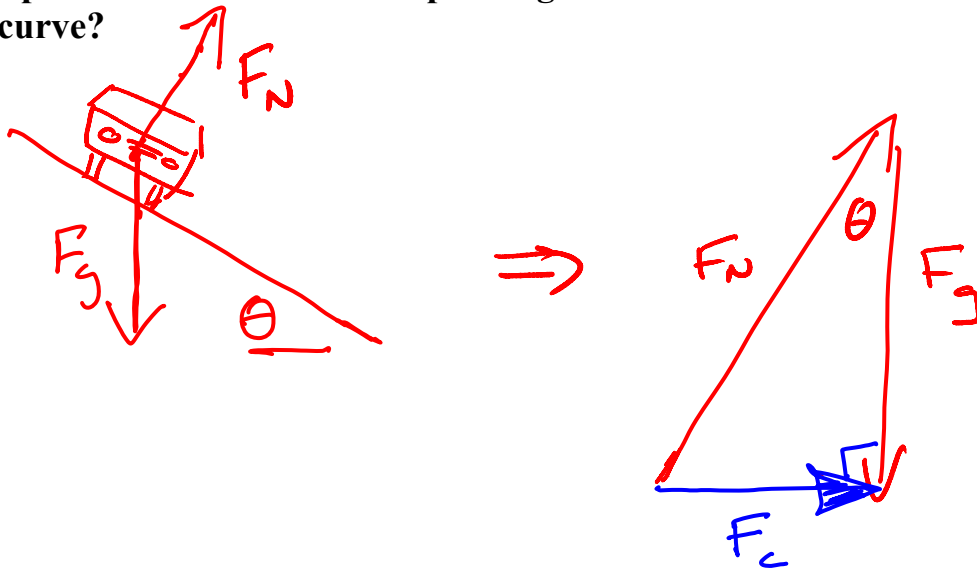
$$\boxed{v = 0.82 \text{ m/s}}$$

$$b) \frac{F_c}{F_T} = \sin \theta \quad F_T = \frac{F_c}{\sin \theta} = \frac{mv^2}{r \sin \theta}$$

$$F_T = \frac{0.140 (.82)^2}{0.12 \sin 30}$$

$$\boxed{F_T = 1.6 \text{ N}}$$

Example #9: A curve of 30 m radius is banked so that a car may make a turn at a speed of 13 m/s without depending on friction at all. What is the slope of the curve?

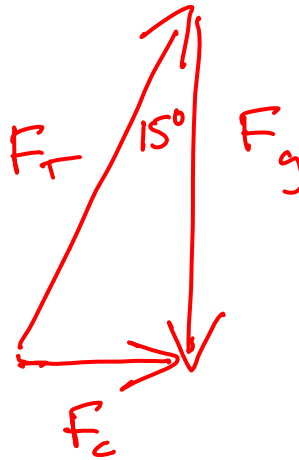
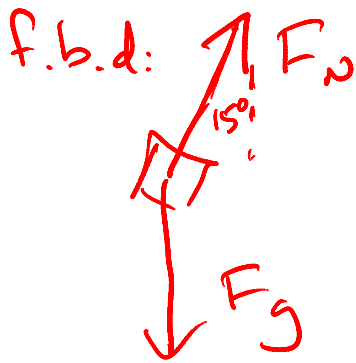


$$\frac{F_c}{F_g} = \tan \theta \quad \frac{mv^2/r}{mg} = \tan \theta$$

$$\frac{13^2}{30(9.8)} = \tan \theta$$

$$\theta = 30^\circ$$

**Example #10:** The radius of a velodrome curve is 40 m and the banked angle is 15°. If  $\mu=0.20$ , what is the maximum speed at which a cyclist can take this curve without slipping?



$$\frac{F_c}{F_g} = \tan \theta$$

$$\frac{mv^2}{r} = mg \tan \theta$$

$$v = \sqrt{40(9.8) \tan 15}$$

$$v = 10. \text{ m/s}$$



